

Non-extremal Black Holes, Harmonic Functions, and Attractor Equations

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ABSTRACT

We present a method which allows to deform extremal black hole solutions into non-extremal solutions, for a large class of supersymmetric and non-supersymmetric Einstein-Vector-Scalar type theories. The deformation is shown to be largely independent of the details of the matter sector. While the line element is dressed with an additional harmonic function, the attractor equations for the scalars remain unmodified in suitable coordinates, and the values of the scalar fields on the outer and inner horizon are obtained from their fixed point values by making specific substitutions for the charges. For a subclass of models, which includes the five-dimensional STU-model, we find explicit solutions.

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1 Introduction

Over the last 15 years there has been tremendous progress in understanding the entropy of extremal black holes in string theory. While the matching of the microscopic entropy [1] and the macroscopic entropy [2] of BPS black holes triggered the ongoing interest in the subject, it has been appreciated more recently that many features of BPS black holes also apply to non-BPS extremal black holes, and, hence do not rely critically on supersymmetry [3, 4]. In contrast, progress on non-extremal solutions has been less impressive. Higher-dimensional non-extremal black hole and black brane solution have been known for some time, as well as non-extremal solutions of compactified supergravity theories [5, 6, 7, 8, 9, 10]. More recently, it has been observed that various non-extremal solutions can be obtained by reducing the equations of motion to first order equations [11, 12, 13, 14, 15, 16, 17]. Treating near-extremal black holes as composites of branes and antibranes accounts for the entropy to leading order, and allows to derive Hawking radiation, including greybody factors [18, 19, 20, 21, 22].

In this article we develop an approach to non-extremal black solutions which keeps the matter sector as general as possible. Our main focus is to get a systematic understanding of how extremal solutions can be made non-extremal, and which features survive this deformation. Much of the success in the study of extremal black holes is due to the good understanding of how they arise as solutions of (super-)gravity in the presence of a generic matter sector. Here ‘generic’ means that the matter sector is as general as allowed by the symmetries underlying the action. The attractor mechanism [2, 23, 24] does not only guarantee that the near-horizon solution, and, hence, the entropy is completely determined by the charges,¹ but also allows to find global black hole solutions in terms of harmonic functions. While solutions cannot always be found in completely explicit form, the field equations can be reduced to a coupled system of algebraic equations, sometimes called ‘generalized stabilization equations’, which express the solution in terms of harmonic functions [26, 27]. The organization of the solution in terms of charges and harmonic functions reflects that from a higher-dimensional (ten- or eleven-dimensional) point of view, black holes are composites of branes and other string or M-theory solitons. This provides the link between black hole thermodynamics and microscopic properties.

One well-known feature of black hole and black brane solutions in various dimensions is that non-extremal solutions differ from extremal ones by the presence of one additional harmonic function, which parametrizes the deviation from extremality. We will review this for the five-dimensional version of the Reissner-Nordstrom solution below. This feature does not only occur for solutions which carry a single type of charge, and thus have a single type of stringy constituent, but also for more complicated solutions, which are multiply charged and can be interpreted as composites of various different types of branes. We interpret this as evidence that the deformation of extremal into non-extremal solutions is ‘universal’, in the sense that it is largely blind to details of the matter sector. Establishing and understanding this in generality is likely to enhance our understanding of non-extremal black holes considerably. In this article we develop an approach based on dimensional reduction over time, harmonic maps and gener-

¹Non-BPS attractors have been studied extensively during the past years, see for example [25] for a review.

alized special geometry. Let us explain these key ingredients and compare them to other approaches taken in the literature.

Dimensional reduction over time, and, for spherically symmetric solutions, dimensional reduction to a one-dimensional problem involving only the radial variable, is a powerful solution generating technique.² It has been applied to Kaluza-Klein black holes [29] and brane-type solutions [30], while in [31] dimensional reduction was used to obtain the black hole attractor equations from the field equations rather than using Killing spinors. More recently, this method has been applied more frequently in the study of extremal non-BPS black holes, and, to some extent, non-extremal black holes [32, 33, 34, 35, 17, 36, 37, 38], and to other brane-type solutions [39]. However, we believe that this method is still under-appreciated, and can become even more powerful if the underlying geometry is fully employed. Dimensional reduction reduces the field equations to the equations of a harmonic map, possibly modified by a potential, from the (reduced) space-time into a scalar target space which encodes all fields contributing to the solution. For static, spherically symmetric solutions one obtains the equation for a geodesic curve in the target space, possibly modified by a potential. The geometry of the reduced space-time reflects the ansatz imposed on the unreduced one. In particular, extremal solutions correspond to flat reduced geometries.³ We will see later that in the non-extremal case the geometry is the time-reduced version of the simplest charged non-extremal solution, the Reissner-Nordstrom solution, independently of the matter content. The geometry of the scalar target space encodes the dynamics of the fields entering into the solution. For supergravity theories the relevant geometries are symmetric spaces for $N > 2$, and various ‘special geometries’ for $N = 2$ supersymmetry. The latter need not be symmetric or even homogeneous spaces, but are characterized by the existence of a potential for the scalar metric. As has become clear recently, there is a more general class of scalar geometries, which might be called ‘generalized special geometries’, which correspond to non-supersymmetric theories, and allow the construction of solutions which share the key features of the solutions of supersymmetric theories [41]. In particular, if one replaces the special real geometry of five-dimensional vector multiplets [42] by the ‘generalized special real geometry’ introduced in [41], then the attractor equations still have the same form discovered in [43, 44] for five-dimensional supergravity, and extremal multi-centered solutions can be obtained in terms of harmonic functions.

In this article we apply this type of approach to the construction of non-extremal solutions. We restrict ourselves to static, spherically symmetric solutions for simplicity. As in [41] we impose that the scalar geometry of the underlying theory, before dimensional reduction, is ‘generalized special real’, and for concreteness we start from five dimensions. This is natural, because generalized special real geometry is a generalization of the special real geometry of five-dimensional vector multiplets. As supersymmetry does not play a role, our results could easily be adapted to any dimension $d \geq 4$ by adjusting numerical parameters.⁴ One limitation which we need to mention is that we only obtain

²We refer to [28] for a review.

³When including Taub-NUT charge, one has to consider more general Ricci-flat geometries [40].

⁴The formulae we use for dimensional reduction contain parameters whose values depend on the number of space-time dimensions. We felt that it was too cumbersome to include this

black hole solutions with electric charges. While this is no restriction in $d > 4$, in $d = 4$ charged black holes can carry both electric and magnetic charge. There is no problem in principle with applying temporal reduction to a four-dimensional theory, but, as is well known from the c -map [45], the isometry group of the resulting scalar manifold is more complicated. Instead of the abelian groups occurring in this paper one obtains solvable Lie groups (of Heisenberg group type). This appears to be a technical rather than conceptual complication, and we have decided to consider the simpler case of abelian isometry groups in this paper, while dyonic solutions are left to future work.

As in [41] our strategy is to simplify the equations of motion until the solution can be expressed in terms of harmonic functions. This is similar in spirit to the way the ‘generalized stabilization equations’ are derived in the framework of the superconformal calculus [27]. An alternative approach is to reduce the equations of motion to first order form, leading to gradient flow equations [12, 32, 33, 15, 16, 17]. This approach mimics the Killing spinor equations of BPS solutions, with the central charge being replaced by a ‘fake superpotential’ which drives the flow. In our approach the re-writing of the field equations in first order form is sidestepped, so that we obtain the solution directly. For the extremal case it was explained in [41] how to obtain the flow equations starting from the harmonic map equation. We expect that this relation can be generalised to cover the results obtained for non-extremal solutions in this paper, but leave a detailed investigation to future work.

Many of the results obtained in the literature are based on the assumption that the scalar target is a symmetric space and exploit the relation to integrability and the Hamilton-Jacobi formalism [34, 35, 17, 36, 37, 38]. Our approach attempts to be less restrictive and only requires the scalar metric to have a potential. Thus roughly speaking we try to work in the analogue of an ‘ $N = 2$ framework’ (special geometry, prepotentials) rather than an ‘ $N > 2$ framework’ (symmetric spaces, integrability). While the explicit non-extremal solutions obtained in this paper happen to correspond to symmetric targets, we argue that the structures which we discover hold more generally, and that the method we are developing is general and flexible enough to deal with target manifolds which are not symmetric spaces. This is supported by the previous observation that extremal multi-centered solutions can be constructed easily for the whole class of models based on generalized special real geometry [41]. Of course, symmetric spaces provide an important and interesting class, and the relation between our approach and the one based on integrability should be clarified in the future.

This paper is organised as follows. In Section 2 we first review the five-dimensional version of the Reissner-Nordstrom solution. Then we perform the reduction of a five-dimensional action based on generalized special real geometry, first with respect to time, then, assuming spherical symmetry, to a one-dimensional effective theory of the radial degrees of freedom. We make some observations which are very helpful in the following: the geometry obtained after reduction over time is, when assuming spherical symmetry, the time-reduced five-dimensional Reissner-Nordstrom metric, irrespective of the matter content. We also identify two useful radial coordinates: the affine curve parameter τ , which is only defined outside the outer horizon, and the isotropic radial coordinate ρ , which allows us to extend solutions up to the inner horizon. After

dependence throughout.

reviewing the relevant background material about generalized special real geometry, we analyze and simplify the remaining equations of motion. We identify a subclass of models, dubbed ‘diagonal’, where solutions can be obtained in closed form. Finding explicit solutions for more general models is left to future work. In Section 3 we lift our solutions to five dimensions and investigate their properties. For diagonal models we obtain non-extremal solutions, valid up to the inner horizon, where all scalar fields are non-constant. The solutions are given in terms of harmonic functions, with one particular function encoding the non-extremality. Extremal solutions are related to non-extremal solutions with the same charges by dressing them in a specific way with the additional harmonic function. In a particular parametrization the expressions for the five-dimensional scalars are identical to the extremal case and solve the same generalized stabilization equations. While there is no attractor or fixed point behaviour in the proper sense, the values of the scalars on the outer and inner horizon are obtained from the fixed point values by specific substitutions, which replace charges by ‘dressed’ charges. Then we turn to a particular diagonal model, the five-dimensional STU-model, which can be obtained (as a subsector) by compactification of type-IIB string theory on $T^4 \times S^1$. We show how our solution is related to the D5–D1 system, and thus establish the relation between our charge parameters and the microscopic charges corresponding to D-branes. Then we turn to the universal solution, which exists in all our models, and show that all five-dimensional scalars are constant, while the metric is the five-dimensional Reissner-Nordstrom metric. Following this we briefly comment on ‘block-diagonal’ models, where the scalar manifold is a product. In this case we obtain solutions where some, but not all scalars can be non-constant. In Section 4 we discuss our results and give an outlook on future research.

2 Dimensional reduction and instanton solutions

2.1 Review of the five-dimensional Reissner-Nordstrom black hole

Some clues how non-extremal, static, spherically symmetric solutions should be approached within the setting of dimensional reduction, harmonic maps, and generalized special geometry can be taken from the five-dimensional version of the Reissner-Nordstrom solution. One standard form of the line element is [5, 6]

$$ds_{(5)}^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4} dt^2 + \left[\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4} \right]^{-1} dr^2 + r^2 d\Omega_{(3)}^2 .$$

In this coordinate system the singularity is located at the origin, $r = 0$, whereas $r_- > 0$ is the inner horizon (Cauchy horizon) and $r_+ > r_-$ is the outer horizon (event horizon). In the extremal limit both horizons coincide, $r_+ = r_-$. Deviations from extremality can be parametrized by the non-extremality parameter $c = \frac{1}{2}(r_+^2 - r_-^2) \geq 0$. For the construction of black hole and black brane solutions one often prefers isotropic coordinates, in which the spatial part of the metric is conformally flat. For the five-dimensional Reissner-Nordstrom solution this is achieved by introducing the new radial coordinate ρ , where

$$\rho^2 = r^2 - r_-^2 .$$

This coordinate system is centered at the inner horizon, which is at $\rho = 0$, while the outer horizon is at $\rho^2 = 2c$. In isotropic coordinates the line element takes the form

$$ds_{(5)}^2 = -\frac{W}{\mathcal{H}^2} dt^2 + \mathcal{H} \left[W^{-1} d\rho^2 + \rho^2 d\Omega_{(3)}^2 \right] . \quad (1)$$

which is parametrized in terms of two harmonic functions⁵

$$\mathcal{H} = 1 + \frac{q}{\rho^2} , \quad W = 1 - \frac{2c}{\rho^2} .$$

The parameter q , which is the electric charge carried by the black hole⁶ is related to r_- by $q := r_-^2$. We prefer to parametrize black holes solutions by the electric charge q and the non-extremality parameter c instead of the positions r_{\pm} of the horizons.

Two interesting limits can be obtained by switching off either of these ‘charges’. Setting $q = 0$ we obtain a five-dimensional version of the Schwarzschild solution, while setting $c = 0$ makes the solution extremal. Thus deforming the solution away from extremality amounts to ‘switching on’ an additional harmonic function in the line element. Experience with supersymmetric solitons in various dimensions suggests that this is a generic feature.

If we perform a dimensional reduction with respect to time, then the four-dimensional (Einstein frame) metric $ds_{(4)}^2$ is related to the five-dimensional (Einstein frame) metric by

$$ds_{(5)}^2 = -e^{2\tilde{\sigma}} dt^2 + e^{-\tilde{\sigma}} ds_{(4)}^2 . \quad (2)$$

For the five-dimensional Reissner-Nordstrom solution the Kaluza-Klein scalar $\tilde{\sigma}$ is given by

$$e^{2\tilde{\sigma}} = \frac{W}{\mathcal{H}^2} .$$

The extremal limit ($W = 1$) has the particular feature that the reduced line element $ds_{(4)}^2$ is flat. As we will see in more detail below, constructing extremal black hole solutions therefore amounts to constructing a harmonic map from a flat manifold (reduced space-time) into a scalar target space, which in Einstein-Maxwell theory accomodates the Kaluza-Klein scalar and the electrostatic potential. The solution corresponds to a null geodesic curve in the scalar target space. Once we consider non-extremal solutions, where $W \neq 1$, the reduced space-time metric $ds_{(4)}^2$ is no longer flat, and the geodesic curve in the scalar target space is no longer null. Our main strategy is to disentangle the non-extremal deformation, which is encoded in the additional harmonic function W , from the degrees of freedom already present in the extremal case.

⁵Here and in the following ‘harmonic function’ refers to a function which is harmonic in the coordinates transverse to the worldline of the black holes (i.e., the four spatial coordinates), with respect to the standard, ‘flat’ Laplacian.

⁶Actually, q is the modulus of the electric charge. Observe that q cannot be negative, as this would introduce additional singularities in the line element. Note that since the energy momentum tensor is quadratic in the Maxwell field strength, the Einstein equations do not ‘see’ the sign of the charge. For convenience, we will refer to q as the electric charge.

2.2 Dimensional reduction

We begin by considering a five-dimensional action of scalars and abelian gauge fields coupled to gravity.

$$\hat{S} = \frac{1}{8\pi G_N^{(5)}} \int d^5 \hat{x} \sqrt{|\hat{g}|} \left[\frac{\hat{R}}{2} - \frac{3}{4} a_{IJ}(h) \partial_{\hat{\mu}} h^I \partial^{\hat{\mu}} h^J - \frac{1}{4} a_{IJ}(h) \hat{F}_{\hat{\mu}\hat{\nu}}^I \hat{F}^{J\hat{\mu}\hat{\nu}} + \dots \right], \quad (3)$$

where $I = 1, \dots, n$ and $\hat{F}_{\hat{\mu}\hat{\nu}}^I = \partial_{\hat{\mu}} \hat{A}_{\hat{\nu}}^I - \partial_{\hat{\nu}} \hat{A}_{\hat{\mu}}^I$.

The dots represent further terms like Chern-Simons and fermionic terms, which could be present, but do not contribute to backgrounds which are static and purely electric. The truncation of five-dimensional supergravity coupled to $n-1$ vector multiplets to such a background has the above form, with a ‘special real’ scalar metric a_{IJ} . This means that the metric has a Hesse potential $\mathcal{V}(h)$,

$$a_{IJ}(h) = \partial_I \partial_J \mathcal{V}(h),$$

and where the Hesse potential takes the special form $\mathcal{V}(h) = -\log \hat{\mathcal{V}}(h)$, with a ‘prepotential’ $\hat{\mathcal{V}}(h)$ which is a homogeneous cubic polynomial. In addition, the scalars must satisfy the hypersurface constraint

$$\hat{\mathcal{V}}(h) = 1. \quad (4)$$

This means that the manifold parametrized by the physical scalar fields is a hypersurface $\hat{M} = \{\hat{\mathcal{V}}(h) = 1\}$ in a Hessian manifold M with metric a_{IJ} . The metric on the hypersurface \hat{M} is the pull-back of a_{IJ} . We will not limit ourselves to supersymmetric theories and allow a larger class of scalar metrics, where the prepotential $\hat{\mathcal{V}}(h)$ is a homogeneous function of arbitrary degree p . Such manifolds might be called ‘generalized special real manifolds’, as they are natural generalizations of the scalar manifolds occurring in supersymmetric theories. The relevant properties of Hessian and (generalized) special real manifolds will be presented in the next section.

We are only interested in five-dimensional solutions which are static and purely electric. In order to construct these solutions we perform a time-like dimensional reduction where we decompose the metric and gauge vectors as follows:⁷

$$\hat{g} = \left(\begin{array}{c|c} -e^{2\tilde{\sigma}} & -e^{2\tilde{\sigma}} \mathcal{A}_{\nu} \\ \hline -e^{2\tilde{\sigma}} \mathcal{A}_{\mu} & e^{-\tilde{\sigma}} (g_{\mu\nu} - e^{2\tilde{\sigma}} \mathcal{A}_{\mu} \mathcal{A}_{\nu}) \end{array} \right), \quad \hat{A}^I = \left(\begin{array}{c} \mathcal{A}_0^I \\ \hline \mathcal{A}_{\mu}^I + \mathcal{A}_0^I \mathcal{A}_{\mu} \end{array} \right).$$

For our class of solutions the Kaluza Klein-vector \mathcal{A}_{μ} vanishes and the last term in the Lagrangian becomes

$$\hat{F}_{\hat{a}\hat{b}}^I \hat{F}^{J\hat{a}\hat{b}} = -2e^{-2\tilde{\sigma}} \partial_{\mu} m^I \partial^{\mu} m^J,$$

where we have made the identification $m^I = \mathcal{A}_0^I$. The resulting four dimensional Euclidean action is

$$S = \frac{1}{8\pi G_N^{(4)}} \int d^4 x \sqrt{|g|} \left[\frac{R}{2} - \frac{3}{4} \partial_{\mu} \tilde{\sigma} \partial^{\mu} \tilde{\sigma} - \frac{3}{4} a_{IJ}(h) \partial_{\mu} h^I \partial^{\mu} h^J + \frac{1}{2} e^{-2\tilde{\sigma}} a_{IJ}(h) \partial_{\mu} m^I \partial^{\mu} m^J + \dots \right]. \quad (5)$$

⁷More details can be found in [46, 41].

As indicated we neglect terms that will not contribute to the type of solution we are interested in. In particular, we neglect four-dimensional gauge fields, because they descend from the magnetic components of the five-dimensional gauge fields. Following the procedure in [41] we make the rescalings

$$h^I = e^{-\tilde{\sigma}} \sigma^I, \quad m^I = \pm \sqrt{\frac{3}{2}} b^I, \quad (6)$$

in order to write the action in the convenient form

$$S = \frac{1}{8\pi G_N^{(4)}} \int d^4x \sqrt{|g|} \left[\frac{R}{2} - \frac{3}{4} a_{IJ}(\sigma) (\partial_\mu \sigma^I \partial^\mu \sigma^J - \partial_\mu b^I \partial^\mu b^J) \right], \quad (7)$$

where we have set $a_{IJ}(\sigma) = e^{-2\tilde{\sigma}} a_{IJ}(h)$ using that a_{IJ} is homogeneous of degree -2 . Similarly, we have

$$\hat{\mathcal{V}}(\sigma) = e^{p\tilde{\sigma}} \hat{\mathcal{V}}(h) = e^{p\tilde{\sigma}}, \quad (8)$$

since the prepotential is homogeneous of degree p .

Note that while the scalars h^I are subject to the constraint (4), the scalars σ^I are unconstrained and combine the $(n-1)$ five-dimensional scalars with the Kaluza-Klein scalar $\tilde{\sigma}$. The scalars σ^I can be interpreted as affine coordinates on an n -dimensional manifold M with Hessian metric $a_{IJ}(\sigma)$. The scalar manifold of the five-dimensional theory is embedded into M as a homogeneous hypersurface \hat{M} . In addition to the σ^I , the four-dimensional theory has n further scalar fields b^I , which descend from the five-dimensional gauge fields. The gauge symmetries of the five-dimensional theory induce n -commuting isometries $b^I \rightarrow b^I + C^I$. The resulting $2n$ scalar manifold N of the four-dimensional theory can therefore be interpreted as the tangent bundle $N = TM$ of M . The Hessian metric of M extends to a split-signature Riemannian metric $a_{IJ}(\sigma) \oplus (-1)a_{IJ}(\sigma)$ on N . It is easy to see that this is a para-Kähler metric⁸ and that the Hesse potential of M is a para-Kähler potential for N [41].

The four-dimensional equations of motion are

$$\frac{1}{\sqrt{|g|}} \partial^\mu \left(\sqrt{|g|} a_{IJ}(\sigma) \partial_\mu \sigma^J \right) - \frac{1}{2} \partial_I a_{JK} (\partial_\mu \sigma^J \partial^\mu \sigma^K - \partial_\mu b^J \partial^\mu b^K) = 0, \quad (9)$$

$$\partial^\mu \left(\sqrt{|g|} a_{IJ}(\sigma) \partial_\mu b^J \right) = 0, \quad (10)$$

$$\begin{aligned} \frac{1}{4} a_{IJ}(\sigma) (\partial_\mu \sigma^I \partial_\nu \sigma^J - \partial_\mu b^I \partial_\nu b^J) - \frac{1}{8} a_{IJ}(\sigma) g_{\mu\nu} (\partial_\gamma \sigma^I \partial^\gamma \sigma^J - \partial_\gamma b^I \partial^\gamma b^J) \\ = \frac{1}{6} R_{\mu\nu} - \frac{1}{12} R g_{\mu\nu}. \end{aligned} \quad (11)$$

The first two equations are the scalar equations of motion. They are equivalent to the geometrical statement that critical points of the action with respect to variation of (σ^I, b^I) define a harmonic map from four-dimensional space-‘time’ (with positive definite metric $g_{\mu\nu}$) into the scalar target manifold N with metric $a_{IJ} \oplus (-1)a_{IJ}$. The third set of equations are Einstein’s equations. They can be simplified by taking the trace of (11) and re-substituting the result back:

$$\frac{1}{4} a_{IJ}(\sigma) (\partial_\mu \sigma^I \partial_\nu \sigma^J - \partial_\mu b^I \partial_\nu b^J) = \frac{1}{6} R_{\mu\nu}. \quad (12)$$

We now impose that the solution is spherically symmetric.⁹ A general spher-

⁸We refer to [58, 46] for a detailed account of para-Kähler geometry.

⁹This type of reduction is frequently used in the literature, see in particular [31, 17].

ically symmetric line element can be written in the form [17]

$$ds_{(4)}^2 = e^{6A(\tau)} d\tau^2 + e^{2A(\tau)} d\Omega_{(3)}^2, \quad (13)$$

where τ is a radial coordinate. The advantage of this parametrization becomes apparent once we look at the reduced equations of motions for the scalar fields:

$$\frac{d}{d\tau} (a_{IJ}(\sigma) \dot{\sigma}^J) - \frac{1}{2} \partial_I a_{JK}(\sigma) (\dot{\sigma}^J \dot{\sigma}^K - \dot{b}^J \dot{b}^K) = 0, \quad (14)$$

$$\frac{d}{d\tau} (a_{IJ}(\sigma) \dot{b}^J) = 0. \quad (15)$$

These are the equations for a geodesic curve on N , written in terms of the coordinates (σ^I, b^I) . For a harmonic map defined on a one-dimensional domain the harmonic equation and the geodesic equation coincide.¹⁰ We observe that the geodesic equation is in *affine form*, which shows that the radial coordinate τ is an affine curve parameter. Other parametrizations of the four-dimensional line element use radial coordinates which are non-affine curve parameters. The reason for τ being an affine parameter is that the Laplace operator for a line element of the form (13) takes the form $\Delta = \frac{\partial^2}{\partial \tau^2} +$ terms independent of τ .

The equations (14) and (15) follow from the variation of the effective action

$$S_{eff} = \int d\tau \frac{1}{4} a_{IJ}(\sigma) (\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J), \quad (16)$$

which is the reduction of (7) in the spherically symmetric background (13).

We still have to reduce the Einstein equations (12). Since we impose spherical symmetry on the scalar fields, the LHS of (12), which is essentially energy momentum tensor, vanishes for all components with $\mu, \nu \neq \tau$. The corresponding components of the Ricci tensor on the RHS of (12) are proportional to $\ddot{A} - 2e^{4A}$, and therefore the Einstein equations imply

$$\ddot{A} - 2e^{4A} = 0. \quad (17)$$

We now consider (12) when $\mu = \nu = \tau$. In this case

$$\frac{1}{4} a_{IJ}(\sigma) (\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J) = \dot{A}^2 - \frac{1}{2} \ddot{A} = c^2, \quad (18)$$

where c^2 is a constant, which we will choose positive below. The fact that $\dot{A}^2 - \frac{1}{2} \ddot{A}$ must be constant follows from (17). We can combine (17) and (18) to get

$$\dot{A}^2 = c^2 + e^{4A}. \quad (19)$$

This first order equation can be solved as follows, for positive c^2 : Taking the square root and multiplying by $-2e^{-2A}$ we find

$$-2\dot{A}e^{-2A} = \pm 2\sqrt{c^2e^{-4A} + 1}.$$

We can then relabel $y(\tau) = e^{-2A(\tau)}$ and hence the equation becomes $\dot{y} = \pm 2\sqrt{c^2y^2 + 1}$. Solving this we find

$$y(\tau) = \frac{\sinh(\pm 2c\tau + D)}{c}.$$

¹⁰In general, the harmonic equation is the trace of the geodesic equation and therefore a weaker condition.

To ensure $y(\tau)$ is positive we choose the positive sign and $D = 0$. We also observe that a negative c^2 would lead to an equation which is solved by trigonometric rather than hyperbolic functions. The resulting solutions are periodic in the radial coordinate and therefore not asymptotically flat. We discard them because we want to construct five-dimensional black holes solutions.¹¹

Thus we find $e^{-2A} = \frac{1}{c} \sinh(2c\tau)$ and our line element is

$$ds_{(4)}^2 = \frac{c^3}{\sinh^3(2c\tau)} d\tau^2 + \frac{c}{\sinh(2c\tau)} d\Omega_{(3)}^2. \quad (20)$$

To see that this is in fact the time reduced Reissner-Nordstrom metric, we replace τ by a new radial coordinate ρ , which is defined by

$$\rho^2 = \frac{ce^{2c\tau}}{\sinh(2c\tau)}. \quad (21)$$

Using this new coordinate, the line element takes the form

$$ds_{(4)}^2 = W^{-\frac{1}{2}} d\rho^2 + W^{\frac{1}{2}} \rho^2 d\Omega_{(3)}^2, \quad (22)$$

where

$$W = 1 - \frac{2c}{\rho^2} = e^{-4c\tau}. \quad (23)$$

To see that this is the time reduced Reissner-Nordstrom metric, we compare the five-dimensional Reissner-Nordstrom metric (1) to the Kaluza-Klein ansatz (2) which relates the five-dimensional to the four-dimensional Einstein frame, and observe that the resulting Euclidean four-dimensional line element is (22). We note that the four-dimensional metric takes this form irrespective of the scalar sector.

From (23) it is manifest that the coordinate τ with range $0 < \tau < \infty$ only covers the range of ρ where $\rho^2 > 2c$. For $0 < \rho^2 < 2c$ the line element (22) becomes imaginary, but looking back at (1) we see that the five-dimensional line element obtained by lifting is real, and that $0 < \rho^2 < 2c$ corresponds to the region between the outer (event) and the inner (Cauchy) horizon. In this region the coordinate t becomes space-like while ρ becomes time-like.¹² It is not surprising that our method, which is based on dimensional reduction over time, does a priori only give us a solution valid outside the event horizon. However, after replacing τ by ρ the analytical continuation to $0 < \rho^2 < 2c$ gives the Reissner-Nordstrom solution up to the inner horizon. Since we have seen that (22) remains unchanged when admitting a more complicated matter sector, we should expect that a similar extension is possible in the presence of non-constant scalar fields. We will come back to this later.

The four-dimensional Einstein equations require that the scalar fields satisfy

$$\frac{1}{4} a_{IJ}(\sigma) \left(\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J \right) = c^2. \quad (24)$$

¹¹If the radial coordinate is analytically continued and becomes timelike, such solutions might correspond to cyclic cosmological solutions.

¹²To be precise, ρ can be continued analytically beyond the event horizon, while t cannot. However, one can introduce a space-like coordinate (which is not the analytical continuation of the coordinate t used outside the horizon), such that the line element takes the form (1) between the outer and the inner horizon [48].

This equation does not follow from the reduced action (16), and must be imposed as a constraint. (It is often called the Hamiltonian constraint, because it descends from the Einstein equations, which are constraints in the Hamiltonian formalism.) Geometrically (24) imposes that the norm of the geodesic vector field (σ^I, b^I) is constant, and is given by the parameter c which appears in the space-time metric. This equation is consistent with (14) and (15), because τ is an affine curve parameter.¹³

While the four-dimensional line element is universal, in the sense that it is independent of the scalar sector, the five-dimensional line element depends on the solution of the scalar field equations through the Kaluza-Klein scalar $\tilde{\sigma}$, which is determined by the four-dimensional scalars through (8). In particular, if the resulting five-dimensional scalars are not constant, then the five-dimensional line element will be different from the five-dimensional Reissner-Nordstrom metric.

We remark that it is very encouraging that the four-dimensional metric is completely determined, and equal to the time-reduced Reissner Nordstrom metric, irrespective of the matter content of the theory. This supports the idea that the deformation of extremal into non-extremal solutions has universal features and can be understood in generality, for ‘arbitrary’ matter content. All features of the solution which depend on the matter sector are encoded in the Kaluza-Klein scalar which is determined by the four-dimensional scalar field equations. Non-extremal solutions differ from extremal solutions through the replacement of the four-dimensional flat metric by the time-reduced Reissner-Nordstrom metric, which is parametrized by a single additional parameter c . Therefore it is reasonable to expect that there is a canonical one-parameter deformation of the harmonic map corresponding to an extremal solution, which deforms a null geodesic in N into a space-like geodesic. This deformation is induced by the deformation of the metric on the domain of the harmonic map from a flat metric to the time-reduced Reissner-Nordstrom metric.

2.3 Hessian manifolds and dual coordinates

In order to solve the remaining equations, we will use the special geometric properties of the target manifold $N = TM$. Since N is completely determined by M , the essential properties are those of the Hessian metric $a_{IJ}(\sigma)$ of M . We now collect the relevant properties of Hessian and (generalized) special real metrics [46, 41].

A Hessian manifold (M, a, ∇) is a manifold M equipped with a pseudo-Riemannian metric a and a flat, torsion-free connection ∇ , such that the third rank tensor ∇a is completely symmetric.¹⁴ In affine coordinates σ^I , where $\nabla_I = \partial_I$, this is equivalent to the statement that $\partial_I a_{JK}$ is completely symmetric. This is the integrability condition for the existence of a Hesse potential for the metric. Thus an equivalent local definition in terms of affine coordinates is that the metric can be written in the form

$$a_{IJ}(\sigma) = \partial_I \partial_J \mathcal{V} = \mathcal{V}_{IJ} , \quad (25)$$

¹³Affine curve parameters are singled out by imposing that the norm of the tangent vector is constant along the curve. This is necessary and sufficient for the geodesic equation to take affine form.

¹⁴The connection ∇ is in general different from the Levi-Civita connection.

where we have introduced the notation $\partial_I \mathcal{V} = \mathcal{V}_I, \dots$. In affine coordinates, the Christoffel symbols of the first kind are completely symmetric and proportional to the third derivatives of the Hesse potential.

For a (generalized) special real metric we impose in addition that the Hesse potential \mathcal{V} has the form

$$\mathcal{V} = -\frac{1}{p} \log \hat{\mathcal{V}}(\sigma) , \quad (26)$$

where the ‘prepotential’ $\hat{\mathcal{V}}$ is a homogeneous function of degree p .¹⁵

$$\hat{\mathcal{V}}(\lambda\sigma^1, \dots, \lambda\sigma^n) = \lambda^p \hat{\mathcal{V}}(\sigma^1, \dots, \sigma^n) . \quad (27)$$

It was shown in [41] that Hesse potentials of this form define four-dimensional models which can be lifted consistently to five-dimensional Einstein-Maxwell-Scalar type theories such as (3).

Using the homogeneity of the prepotential we deduce that

$$\hat{\mathcal{V}}_I(\sigma)\sigma^I = p\hat{\mathcal{V}}(\sigma) , \quad (28)$$

and differentiation implies

$$\hat{\mathcal{V}}_{IJ}\sigma^I = (p-1)\hat{\mathcal{V}}_J . \quad (29)$$

If we write the metric in terms of the prepotential

$$a_{IJ}(\sigma) = \mathcal{V}_{IJ} = -\frac{1}{p} \left(\frac{\hat{\mathcal{V}}_{IJ}}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2} \right) , \quad (30)$$

we can use (28) and (29) to deduce that

$$a_{IJ}\sigma^J = -\mathcal{V}_I . \quad (31)$$

It follows that contracting the coordinates with the metric we are left with unity:

$$a_{IJ}\sigma^I\sigma^J = 1 . \quad (32)$$

It is important to note that this is not a constraint on the coordinates σ^I but an identity which follows from the particular form (26) of the Hesse potential. As is evident from (30) the metric coefficients a_{IJ} are homogeneous of degree -2 . Thus the metric (as a tensor) is homogeneous of degree 0. As a consequence, re-scalings $\sigma^I \rightarrow \lambda\sigma^I$ of the affine coordinates act as isometries on M , and also on $N = TM$. This additional symmetry will be helpful in solving the equations of motion.

We now motivate the introduction of dual coordinates by first noting that the equation of motion (14) simplifies if we can find dual coordinates σ_I which satisfy

$$\dot{\sigma}_I = a_{IJ}(\sigma)\dot{\sigma}^J . \quad (33)$$

For extremal black holes, where $c = 0$, this allows one immediately to express the solution in terms of harmonic functions, even if no spherical symmetry is imposed [41]. If a_{IJ} is Hessian, then dual coordinates can always be found

¹⁵For the special real metrics of five-dimensional supersymmetric theories, $p = 3$, and $\hat{\mathcal{V}}$ must be a polynomial.

explicitly and are given by $\sigma_I \propto \mathcal{V}_I$. From the identity (31) we see that these coordinates can be written as

$$\sigma_I = -a_{IJ}\sigma^J . \quad (34)$$

The minus sign might be counter-intuitive, but one should remember that the σ^I are functions (local coordinates) and not vector fields. The dual coordinates σ_I are algebraic functions of the affine coordinates σ^I .

For example, if the prepotential is a general homogeneous polynomial $\hat{\mathcal{V}} = C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}$ of degree p , then dual coordinates are given by

$$\sigma_I = -\frac{1}{p} \frac{\partial_I C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}}{C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}} . \quad (35)$$

A special case of particular interest is if the prepotential is of the form $\hat{\mathcal{V}} = \sigma^1 \dots \sigma^p$ in which case dual coordinate are

$$\sigma_I = -\frac{1}{p} \frac{1}{\sigma^I} . \quad (36)$$

While it is always possible to find explicit expressions for the dual coordinates in terms of the affine coordinates σ^I , inverting this relation amounts to solving n coupled algebraic equations, which in general cannot be done in closed form. Solving these equations is in fact equivalent to solving the (five-dimensional) black hole attractor equations [41].

2.4 Four-dimensional instanton solutions

We now proceed to solving the equations of motion (14), (15) and (24). Since they were derived from the action of a Euclidean non-linear sigma model, the solutions will be referred to as instantons. We will consider Hessian manifolds of the form (26) and we will formulate the solutions in terms of the dual coordinates, making use of the identities derived in the previous section.

The equations of motion (15) for the axions b^I are solved by

$$a_{IJ}(\sigma) \dot{b}^I = \tilde{q}_I = \text{const.} , \quad (37)$$

where \tilde{q}_I are the ‘axion charges’ (or ‘instanton charges’), which are the conserved charges corresponding to the isometries $b^I \rightarrow b^I + C^I$.

Now we turn our attention to (14). Using the dual coordinate σ_I , this becomes

$$\ddot{\sigma}_I - \frac{1}{2} \partial_I a_{JK}(\sigma) (\dot{\sigma}^J \dot{\sigma}^K - \dot{b}^J \dot{b}^K) = 0 , \quad (38)$$

and using that $\partial_I a_{JK} = -a_{JL} a_{KM} \partial_I a^{LM}$ this can be written as

$$\ddot{\sigma}_I - \frac{1}{2} \partial_I a^{JK}(\sigma) (\dot{\sigma}_J \dot{\sigma}_K - \tilde{q}_J \tilde{q}_K) = 0 , \quad (39)$$

In the extremal case, where the geodesic curve on N is null,¹⁶ the second term is absent, and the equations collapse to $\ddot{\sigma}_I = 0$, which is solved by

$$\sigma_I(\tau) = A_I + B_I \tau .$$

¹⁶To be precise, the geodesic curve corresponding to an extremal solution is not only null, but satisfies $\dot{\sigma}^I = \pm \dot{b}^I$. See [46, 41] for an interpretation in terms of the para-Kähler geometry of N .

In the extremal case the standard radial coordinate (centered at the horizon) is ρ , where $\rho^2 = \frac{1}{2\tau}$, so that ¹⁷

$$\sigma_I(\rho) = A_I + \frac{2B_I}{\rho^2} .$$

Thus the solution can be expressed in terms of n spherically symmetric harmonic functions, which depend on $2n$ parameters. For $c \neq 0$ the equation (39) is more complicated and involves the Christoffel symbols of N . To simplify the problem, we contract (39) with σ^I , to obtain a single equation. This leads to an enormous simplification, provided that we make full use of the special properties of the scalar metric. Since the metric is homogeneous of degree -2 , we have

$$\sigma^I \partial_I a_{JK} = -2a_{JK} .$$

Combining this with (34), the contracted equations reduces to

$$a^{IJ} \sigma_I \ddot{\sigma}_J = 4c^2 . \quad (40)$$

Comparing to the Hessian identity (32) we see that this equation implies that

$$4c^2 \sigma_I = \ddot{\sigma}_I + X_I , \quad (41)$$

where X_I vanishes when contracted with σ^I , $\sigma^I X_I = 0$. One obvious strategy is to look for solutions where $X_I = 0$. In this case the equations reduce to the linear equations

$$4c^2 \sigma_I = \ddot{\sigma}_I , \quad (42)$$

which are elementary to solve. We can write the general solution as

$$\sigma_I = A_I \cosh 2c\tau + \frac{1}{2c} B_I \sinh 2c\tau , \quad (43)$$

where we have chosen the appropriate factors so that in the extremal limit

$$\sigma_I \xrightarrow[c \rightarrow 0]{} A_I + B_I \tau . \quad (44)$$

The solution contains $2n$ arbitrary constants, which is as many as we expect for the general solution of the original equation (38). However, we have assumed without justification that $X_I = 0$, and therefore we still have to investigate whether (43) is a solution, or even the general solution, of (38). Therefore we substitute (43) back into (38). Using $\ddot{\sigma}_I = 4c^2 \sigma_I$, together with

$$\sigma_I = -a_{IJ} \sigma^J = \frac{1}{2} \sigma^K \partial_K a_{IJ} \sigma_J = \frac{1}{2} \partial_I a_{JK} \sigma^J \sigma^K = -\frac{1}{2} \partial_I a^{JK} \sigma_J \sigma_K ,$$

which combines various of the special identities satisfied by a_{IJ} , we obtain

$$\partial_I a^{JK} (4c^2 A_J A_K - B_J B_K + \tilde{q}_J \tilde{q}_K) = 0 . \quad (45)$$

This equation is to be viewed as an algebraic constraint on the integration constants A_I and B_I . Since we assume that the solution for σ_I is given by (43),

¹⁷Affine coordinates are only unique up to affine transformations. The normalization has been chosen for later convenience.

the ‘Christoffel symbols’ $\partial_I a^{JK}$ are functions of the integration constants A_I , B_I and of the curve parameter τ . Thus we obtain n algebraic relations between the $3n$ constants A_I , B_I and \tilde{q}_I which have to be satisfied along the geodesic curve, i.e. for all values of the curve parameter τ . These conditions are hard to investigate without specifying the scalar metric a_{IJ} explicitly. However, we will prove the following three statements in the following sections:

1. If the metric a_{IJ} and the Christoffel symbols are diagonal (or can be brought to diagonal form by a linear transformation of the affine coordinates σ^I), then (43), with $2n$ independent constants A_I, B_I is the general solution. In this case the metric of the scalar manifold N is the product of n two-dimensional metrics, and the scalars σ_I completely decouple from one another. In the resulting solution all scalars σ_I are independent, in the sense that all mutual ratios are non-constant, and the corresponding five-dimensional scalars are non-constant. The Reissner-Nordstrom solution is recovered by taking the five-dimensional scalars to be constant, which is equivalent to taking all four-dimensional scalars to be proportional to one another.
2. For arbitrary a_{IJ} there is always a solution of the form (43) depending on $n + 1$ independent parameters, which can be taken to be the charges \tilde{q}_I and the non-extremality parameter c . For these solutions the four-dimensional scalar fields σ_I are proportional to one another, and the five-dimensional scalars are constant. The metric is the five-dimensional Reissner-Nordstrom metric. These solutions are therefore non-extremal deformations of ‘double extreme’ five-dimensional black holes, which are extremal black holes with constant (five-dimensional) scalars. This result is not unexpected, but reassuring, because it shows us how to recover the non-extremal Reissner-Nordstrom solution, with the slight generalization that we have n independent gauge fields and thus n independent charges. We call this solution, which can be found for all models, the universal solution.
3. If the metric and the Christoffel symbols are block diagonal, with $1 < k < n$ blocks, or if they can be brought to this form by a linear transformation of the affine coordinates σ^I , then we obtain solutions of the form (43) with $n + k$ independent integration constants. In this case only the ratios between four-dimensional scalars which belong to the same block have to be constant, and the five-dimensional solutions have $k - 1$ parameters which correspond to changing the values of the scalars at infinity. Such block diagonal models provide intermediate cases between the diagonal models $k = n$ and the generic models where $k = 1$.

We stress that we of course expect that the general solution always has $2n$ independent integration constants, irrespective of the form of the scalar metric. However, solutions of the form (43), which are obtained by assuming $X_I = 0$, seem only to account for a subset of solutions if the Christoffel symbols are not diagonalizable. The study of more general solutions is left to future work.

3 Dimensional Lifting and black hole solutions

We now proceed to discuss the three cases in turn.

3.1 The general solution for diagonal models

Instead of solving (45), we can impose the stronger condition

$$4c^2 A_J A_K - B_J B_K + \tilde{q}_J \tilde{q}_K = 0 . \quad (46)$$

If we do not make assumptions on the structure of a_{IJ} , this has to be true for all values of J, K , in order to solve (45). This imposes severe constraints on the constants A_I, B_I , which, in general, only allows solutions where all four-dimensional scalars are proportional to one another. This solution, which we call the universal solution, will be discussed in the next section.

In this section we will restrict the scalar metric in such a way that we obtain the general solution. Specifically, we assume that $\partial_I a^{JK} = 0$ for $J \neq K$. Such models will be referred to as diagonal models in the following. For diagonal models (45) is already solved if we impose (46) for $J = K$:

$$4c^2 A_J^2 - B_J^2 + \tilde{q}_J^2 = 0 . \quad (47)$$

This equation can be solved explicitly for the A_I , or for the B_I , or for any linear combinations thereof, in terms of the charges \tilde{q}_I and of the remaining n independent combinations of the A_I and B_I . In the following it is convenient to consider A_I and B_I as independent parameters and to compute the resulting charges \tilde{q}_I from (47):

$$\tilde{q}_J^2 = B_J^2 - 4c^2 A_J^2 . \quad (48)$$

In order to bring the solution to a form suitable for dimensional lifting and interpretation as a black hole solution, we remember that the four-dimensional Euclidean line element takes the form of the time-reduced five-dimensional Reissner-Nordstrom metric (20), irrespective of the details of the matter sector. Therefore it is natural to replace the radial coordinate τ , which is an affine parameter for curve in N corresponding to the solution, by the standard radial coordinate (21):

$$\rho^2 = \frac{ce^{2c\tau}}{\sinh(2c\tau)} .$$

Observe that in the extremal limit $c \rightarrow 0$ we recover the relation

$$\rho^2 = \frac{1}{2\tau} . \quad (49)$$

It is useful to note that

$$\sigma_I = \frac{1}{2e^{-2c\tau}} \left(A_I(1 + e^{-4c\tau}) + \frac{1}{2c} B_I(1 - e^{-4c\tau}) \right) .$$

As discussed earlier the non-extremal Reissner-Nordstrom solution is obtained from the extremal one through dressing the line element by the additional harmonic function

$$W(\rho) = 1 - \frac{2c}{\rho^2} = e^{-4c\tau} .$$

We now observe that

$$\sigma_I(\rho) = \frac{H_I(\rho)}{W(\rho)^{1/2}} ,$$

where

$$H_I(\rho) = A_I + \frac{B_I - 2cA_I}{2\rho^2}$$

are harmonic functions. Since the extremal solution is given by [41]

$$\sigma_I^{(\text{extr})} = H_I(\rho) = A_I + \frac{q_I}{\rho^2} ,$$

with constants A_I and q_I , we see that the non-extremal solution is obtained from the extremal one by dressing the solution by the additional factor $W^{1/2}(\rho)$. In addition, the relation between the standard radial coordinate ρ and the affine parameter τ depends on c according to (21). The constants A_I encode the values of the dual scalars infinity, and are independent of c :

$$A_I = \sigma_I(\rho \rightarrow \infty) .$$

The constants B_I and q_I are related to one another and to the charges \tilde{q}_I . In the extremal limit they only differ by constant factors, and their relation is independent of the A_I :

$$c = 0 \Rightarrow q_I = \frac{1}{2}B_I = \pm \frac{1}{2}\tilde{q}_I .$$

For non-extremal solutions the relations between these three sets of quantities depend on c and on the A_I according to (48) and

$$q_I = \frac{1}{2}(B_I - 2cA_I) .$$

Note that a relation of this form is precisely what we should expect from the extremal limit of the general solution (44) with the change of variables (49). Given these identifications, and using the radial coordinate ρ , the relation between the non-extremal and extremal solution is given by

$$\sigma_I = \frac{H_I}{W^{1/2}} \xrightarrow{c \rightarrow 0} H_I = \sigma_I^{(\text{extr})} ,$$

where $H_I(\rho)$ and $W(\rho)$ are spherically symmetric harmonic functions in four dimensions.

Our solution depends on $2n+1$ independent parameters: the values A_I of the scalars at infinity, the non-extremality parameter c and the instanton charges \tilde{q}_I . Instead of the charges \tilde{q}_I we could use alternatively the integration constants B_I or q_I . So far the charges \tilde{q}_I are the most natural choice, as they have a direct physical interpretation as the conserved charges associated with the axionic shift symmetries. In the extremal limit, the B_I and q_I become proportional to the charges \tilde{q}_I , but in the non-extremal case their relation to the \tilde{q}_I is a function of c and depends on the values A_I of the scalars at infinity. Below we will see that q_I have a physical interpretation from the five-dimensional point of view.

We can lift our solution to five dimensions and control the extremal limit. Since $\sigma^I = -a^{IJ}(\sigma)\sigma_J$, it suggests itself to define functions H^I by

$$\sigma^I = W^{1/2}H^I .$$

Note that $H^I H_I = \sigma^I \sigma_I = 1$, and due to the scaling properties of the metric we have

$$H^I = -a^{IJ}(H) H_J .$$

While the H_I are harmonic functions, the H^I are not. However, since the extremal solution is given by $\sigma_I^{(\text{extr})} = H_I$, the H^I are the solutions for the scalars σ^I in the extremal limit, $\sigma_{(\text{extr})}^I = H^I$. Thus the above rescaling allows us to write the non-extremal solution as a rescaled version of the extremal one, both in terms of the scalars σ^I and the dual scalars σ_I .

We now use that the four-dimensional Euclidean metric is (22)

$$ds_4^2 = W^{-1/2} d\rho^2 + W^{1/2} \rho^2 d\Omega^2 ,$$

and that the four- and five-dimensional line elements are related by (2)

$$ds_5^2 = -e^{2\tilde{\sigma}} dt^2 + e^{-\tilde{\sigma}} ds_4^2 ,$$

where the Kaluza-Klein scalar $\tilde{\sigma}$ is given in terms of the four-dimensional scalars by

$$e^{p\tilde{\sigma}} = \hat{\mathcal{V}}(\sigma) = W^{p/2} \hat{\mathcal{V}}(H) .$$

Therefore the five-dimensional line element takes the form

$$\begin{aligned} ds_5^2 &= -W \hat{\mathcal{V}}(H)^{2/p} dt^2 + \frac{1}{W^{1/2} \hat{\mathcal{V}}^{1/p}(H)} \left(\frac{d\rho^2}{W^{1/2}} + W^{1/2} \rho^2 d\Omega^2 \right) \\ &= -W \hat{\mathcal{V}}(H)^{2/p} dt^2 + \frac{1}{\hat{\mathcal{V}}(H)^{1/p}} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega^2 \right) . \end{aligned}$$

We observe that the five-dimensional Reissner-Nordstrom metric is recovered if

$$\hat{\mathcal{V}}(H)^{1/p} = \frac{1}{\mathcal{H}} ,$$

where $\mathcal{H} = 1 + \frac{q}{\rho^2}$. We will see below how this arises as a particular limit of general solution for diagonal models.

Remember that we have obtained the general solution by making the assumption that the model is ‘diagonal’, in the sense that the Christoffel symbols $\partial_I a^{JK}$ are diagonal in J, K for all I . One class of prepotentials which leads to such models is

$$\hat{\mathcal{V}}(\sigma) = \sigma^1 \sigma^2 \dots \sigma^p .$$

For $p = 3$ we recover the five-dimensional STU model, while for $p > 3$ the resulting models are not supersymmetric, but have properties similar to the STU models as far as black hole solutions are concerned [46, 41]. The scalar manifolds N of the four-dimensional models obtained by reduction over time are of the form

$$N = \left(\frac{SU(1,1)}{SO(1,1)} \right)^p .$$

For $p = 3$ we obtain the Euclidean version of the four-dimensional STU model [46, 41].

With this choice of prepotential the dual coordinates are

$$\sigma_I = \frac{1}{\sigma^I} .$$

This can be solved for the original scalars σ^I , so that we obtain the solution in closed form:

$$\sigma^I = \frac{W^{1/2}}{H_I} .$$

Therefore

$$\hat{\mathcal{V}}(\sigma) = W^{p/2} (H_1 \cdots H_p)^{-1} ,$$

and the resulting five-dimensional line element is

$$ds_{(5)}^2 = -\frac{W}{(H_1 \cdots H_p)^{2/p}} dt^2 + (H_1 \cdots H_p)^{1/p} \left(\frac{d\rho^2}{W} + \rho^2 d\Omega^2 \right) .$$

The non-extremal five-dimensional Reissner-Nordstrom metric is obtained in the special case where all the harmonic functions H_I are proportional to one another:¹⁸

$$H_1 \propto H_2 \propto \cdots H_p \propto \mathcal{H} = 1 + \frac{q}{\rho^2} ,$$

so that

$$H_1 \cdots H_p = \mathcal{H}^p ,$$

and

$$ds_{(5)}^2 = -\frac{W}{\mathcal{H}^2} dt^2 + \mathcal{H} \left[W^{-1} d\rho^2 + \rho^2 d\Omega_{(3)}^2 \right] .$$

We can also find explicit expressions for the five-dimensional scalars. Remember that (6)

$$h^I = e^{-\tilde{\sigma}} \sigma^I .$$

Therefore

$$h^I = \hat{\mathcal{V}}(\sigma)^{-1/p} \sigma^I = \hat{\mathcal{V}}(H)^{-1/p} H^I = \frac{(H_1 \cdots H_p)^{1/p}}{H_I} . \quad (50)$$

We observe that W has cancelled out so that we obtain the same solution for h^I as in the extremal case [41]. Taking all harmonic functions to be proportional to one another amounts to taking the five-dimensional scalars to be constant. In this case the metric takes the Reissner-Nordstrom form, as it must. The only difference between this solution and Reissner-Nordstrom solution of five-dimensional Einstein Maxwell theory is that our solutions are charged under an arbitrary number n of abelian gauge fields.

Our observation that the solution for the five-dimensional scalars remains the same as in the extremal case raises the question what happens to the attractor mechanism. As we have discussed previously, the function W changes sign at $\rho^2 = 2c$, and the four-dimensional metric (22) becomes imaginary. For the Reissner-Nordstrom solution, which we recover by taking all harmonic functions to be proportional, this corresponds to crossing the outer horizon into the region where the coordinate ρ becomes time-like. While our construction of the solutions via dimensional reduction over time is a priori only valid for $\rho^2 > 2c$, we know that the Reissner-Nordstrom solution is obtained by continuing the solution to $0 < \rho^2 < 2c$ and lifting. Since our general solution can be viewed as deforming the Reissner-Nordstrom solution by turning non-constant scalar

¹⁸The overall normalization of \mathcal{H} is fixed by imposing that the five-dimensional line element approaches the standard line element of five-dimensional Minkowski space.

fields, we should expect that the general solution remains valid too. To show this we need to make the assumption that $\hat{\mathcal{V}}(H) \neq 0$ for $\rho^2 > 0$ to exclude additional singularities of the line element. In the extremal case it is well known that such singularities are related to scalars field running off to infinity on \hat{M} , or approaching a singular locus of \hat{M} [49]. This behaviour can be avoided by choosing suitable initial conditions for the scalar fields at infinity. In particular, as long we stay ‘close enough’ to the Reissner-Nordstrom solution, no additional singularity can arise. Then the outer and inner horizon are still encoded in W and located at $\rho^2 = 2c$ and $\rho = 0$, respectively. Note that while (22) becomes imaginary at $\rho^2 = 2c$, the resulting five-dimensional remains real because the Kaluza Klein exponential

$$e^{\tilde{\sigma}} = W^{1/2} \hat{\mathcal{V}}(H)^{1/p}$$

becomes imaginary, too.¹⁹ The overall effect on the five-dimensional line element is that ρ becomes time-like while t becomes space-like. We also observe that the solution (50) for the five-dimensional scalars is real (and analytical) for $\rho > 0$. Therefore it makes sense to consider the limit $\rho \rightarrow 0$, which now corresponds to the inner horizon. We find that the scalars formally exhibit fixed point behaviour, in the sense that the solution only depends on the charges q_I , and becomes independent of the remaining constants A_I :

$$h^I \xrightarrow[\rho \rightarrow 0]{} \frac{(q_1 \cdots q_p)^{1/p}}{q_I}.$$

However, we need to remember that the parameters q_I are different from the electric charges \tilde{q}_I . In order to get a better understanding, let us consider the interpretation of the various parameters from the five-dimensional point of view. The \tilde{q}_I are, up to normalization, the electric charges of the black hole, i.e. the Noether charges associated with the conserved current $j_{I|\hat{\nu}} = \partial^{\hat{\mu}}(a_{IJ}(h)F_{\hat{\mu}\hat{\nu}}^J)$. The parameters A_I are the values of the four-dimensional dual scalars σ_I at infinity. In five dimensions, these degrees of freedom reorganise themselves into $n - 1$ scalars and one degree of freedom residing in the metric. In our parametrization, the n scalars h^I are subject to the constraint $\hat{\mathcal{V}}(h) = 1$, and the Kaluza-Klein scalar is given by $e^{p\tilde{\sigma}} = \hat{\mathcal{V}}(\sigma)$. For a five-dimensional black hole solution we should normalize the metric such that it approaches the five-dimensional Minkowski metric at infinity:²⁰

$$e^{\tilde{\sigma}} \xrightarrow[\rho \rightarrow \infty]{} 1.$$

This imposes one constraint between the constants A_I which reflects that there are only $n - 1$ five-dimensional scalars for which we can choose asymptotic values. Thus the five-dimensional solution only depends on $2n$ independent parameters. The additional parameter which we gain by dimensional reduction can be interpreted as the size of the dimension we reduce over, or, equivalently,

¹⁹Here we regard $e^{\tilde{\sigma}}$ as a function that becomes imaginary when continued to $\rho^2 < 2c$. A more systematic approach would be to replace $\tilde{\sigma}$ by a new variable. Since $\tilde{\sigma}$ is defined as the Kaluza Klein scalar for time-like reduction, it is clear that a different variable should be introduced when the reduced dimension becomes spacelike. However, we leave a more detailed investigation of the region between horizons to future work.

²⁰Changing this normalization by a constant factor amounts to rescaling the five-dimensional Newton constant.

as the ratio between the five-dimensional and four-dimensional Newton constant, since

$$\frac{1}{G_N^{(4)}} = \frac{1}{G_N^{(5)}} \int_0^{2\pi R} dt \sqrt{|g_{tt}|} = \frac{1}{G_N^{(5)}} 2\pi R e^{\tilde{\sigma}(\infty)} .$$

While we can use natural units and set $G_N^{(5)} = \frac{1}{16\pi}$, the ratio of $G_N^{(5)}$ and $G_N^{(4)}$ becomes a physical parameter once we reduce. However this parameter is irrelevant as far as five-dimensional black holes are concerned.

The parameters q_I arise as integration constants for the solution when using the coordinate ρ . Their relation to the electric charges depends on c and the asymptotic scalar fields through

$$2q_I = \sqrt{\tilde{q}_I + 4cA_I^2} - 2cA_I .$$

From the five-dimensional point of view the q_I have a direct physical interpretation because they determine the asymptotics of the five-dimensional scalars at the inner horizon.

Since A_I (subject to one constraint), c , and q_I are a set of $2n$ independent parameters, one might say that we have fixed point behaviour at the inner horizon in the sense that the scalars become independent of A_I and c and are completely determined by the q_I . While this is formally correct, it is more natural to consider A_I (subject to one constraint), c , and the charges \tilde{q}_I as the independent parameters. Then the asymptotic values of the scalars at the inner horizon do depend on their values at infinity, and on c , in addition to the charges, but only through n independent combinations $q_I = q_I(\tilde{q}_I, A_I, c)$. One might call this a ‘dressed attractor’, or ‘dressed fixed point’.²¹ In the extremal limit q_I and \tilde{q}_I become proportional and the usual attractor behaviour is recovered.

In the extremal case the asymptotic metric at the event horizon is of Bertotti-Robinson type, hence a product of maximally symmetric spaces and therefore an alternative ground state. This is not the case for non-extremal black holes. We also note that the metric at the inner horizon has a two-fold dependence on parameters other than the charges \tilde{q}_I : First it depends on c and A_I through the q_I , second it acquires an additional universal dependence on c through the additional harmonic function W .

Having identified $A_I = \sigma_I(\infty)$ and \tilde{q}_I or q_I as the physical parameters, let us summarize the relation between the charges \tilde{q}_I , which are the electrical charges as defined by current conservation in (super-)gravity and the charges q_I which govern the asymptotics on the inner horizon,

$$\tilde{q}_I = 2\sqrt{q_I^2 + 2cq_I\sigma_I(\infty)} ,$$

and the inverse relation:

$$q_I = \frac{1}{2}\sqrt{\tilde{q}_I^2 + 4c^2\sigma_I(\infty)^2 - c\sigma_I(\infty)} .$$

We now turn our attention to the outer horizon, which is located at $\rho = \sqrt{2c}$. On the outer horizon the harmonic functions H_I take the values

$$H_I = \frac{\bar{q}_I}{2c} ,$$

²¹We refrain from calling this a ‘fake attractor.’

where the \bar{q}_I bare a striking relationship to the dressed charges q_I of the inner horizon

$$\bar{q}_I = \frac{1}{2} \sqrt{\tilde{q}_I^2 + 4c^2 \sigma_I(\infty)^2 + c \sigma_I(\infty)} .$$

Inspection of the scalar fields on the outer horizon reveals the limit²²

$$h^I \xrightarrow[\rho \rightarrow \sqrt{2}c]{} \frac{(\bar{q}_1 \cdots \bar{q}_p)^{1/p}}{\bar{q}_I} .$$

We can interpret the \bar{q}_I as dressed charges which determine the values of the scalars on the outer horizon. In this sense they exhibit similar ‘dressed attractor’ behaviour on the outer horizon as on the inner horizon. In particular, we observe formally the same fixed point behaviour in the extremal limit. Indeed, this must be the case as in the extremal limit the outer and inner horizons coincide. The dressed charges on the inner and outer horizon are related to the electric charges through

$$\tilde{q}_I^2 = 4q_I \bar{q}_I .$$

While the ‘dressed attractor’ behaviour is not attractor behaviour in the proper sense, it demonstrates that the functional dependence of the solution on the integration constants is not generic, but takes a restricted form. This is what one should expect if the field equations can be reduced to first order form.

One important feature of non-extremal charged solutions is that the coordinate ρ becomes time-like at the outer horizon. Therefore the flow becomes a flow in time rather than in space between the outer and inner horizon. This should have interesting implications for time-dependent solutions and in particular cosmology, since the between horizon region of non-extremal black holes is a natural starting point for the construction of (S-brane type) cosmological solutions [51, 52]. A related question is whether something can be learned about the time evolution of non-extremal black holes, which are expected to loose mass through Hawking radiation and to approach the extremal limit.

In the context of string theory, supergravity provides the macroscopic (= long wavelength = low energy) description of black holes. For some types of black holes string theory provides a microscopic description of black holes in terms of strings, D-branes, and other string solitons. While extremal black holes correspond to ground states of brane configurations, non-extremal black holes correspond to excited states. Since our class of solutions contains the five-dimensional STU model, which occurs as a subsector in various string compactifications, it is natural to use these models to investigate the microscopic interpretation of our solutions.

3.2 The STU model and IIB string theory on T^5

The five-dimensional STU model is based on the Hesse potential

$$\mathcal{V} = -\log(\sigma^1 \sigma^2 \sigma^3) = -\log \sigma^1 - \log \sigma^2 - \log \sigma^3 .$$

It describes two vector multiplets coupled to supergravity and arises (together with hypermultiplets which can be truncated out consistently) as the classical

²²For completeness we remark that a similar relation holds at radius $\rho = \sqrt{c}$, with \bar{q}_I replaced by the integration constants B_I .

limit of the compactification of the heterotic string on $K3 \times S^1$ with instanton numbers $(12 - n, 12 + n)$, where $n = 0, 1, 2$ [53]. Furthermore, it arises as a universal subsector in compactifications with $N = 4$ and $N = 8$ supersymmetry, in particular in type-II compactifications on T^5 , as reviewed in [47]. Let us first collect the relevant formulae: The five-dimensional line element is given as

$$\begin{aligned} ds_{(5)}^2 &= -e^{2\tilde{\sigma}} dt^2 + e^{-\tilde{\sigma}} ds_{(4)}^2 \\ &= -\frac{W}{(H_1 H_2 H_3)^{2/3}} dt^2 + (H_1 H_2 H_3)^{1/3} \left[W^{-1} d\rho^2 + \rho^2 d\Omega_{(3)}^2 \right], \end{aligned}$$

and the five-dimensional scalars h^I are given by

$$h^I = e^{-\tilde{\sigma}} \sigma^I = \left(\frac{H_J H_K}{H_I^2} \right)^{1/3},$$

where I, J, K are pairwise distinct. The limit on the inner horizon is:

$$h^I \xrightarrow{\rho \rightarrow 0} \left(\frac{q_J q_K}{q_I^2} \right)^{1/3}.$$

The same solution was found in [21, 47], using the results of [50], by compactification of the type-IIB string theory on T^5 . One particular realization is a system which carries integer D1-brane charge Q_1 , integer D5-brane charge Q_5 and integer momentum N along the D1-brane. These charges can be expressed in terms of the string coupling g , the radii R_5, \dots, R_9 , the non-extremality parameter c and three ‘boost parameters’²³ $\alpha_1, \alpha_5, \alpha_N$ as follows:

$$Q_1 = \frac{V}{g} c \sinh(2\alpha_1), \quad Q_5 = \frac{1}{g} c \sinh(2\alpha_5), \quad Q_N = \frac{R^2 V}{g} c \sinh(2\alpha_N),$$

where $V = R_5 R_6 R_7 R_8$ and $R = R_9$. Since the underlying brane configuration consists of D1 branes oriented along the x_9 direction within the D5 world volume, the moduli are the radius $R = R_9$, the volume V of the torus spanned by the other four compact directions, and the string coupling.

In [47] the extremal limit is performed by sending $c \rightarrow 0$, and the boost parameters $\alpha_I \rightarrow \infty$, while keeping the brane charges Q_I and the moduli g, R, V constant.

To relate this to our solutions, we note that harmonic functions in [47] take the form

$$H_I = 1 + \frac{2c \sinh \alpha_I}{\rho^2}.$$

Matching this with our parametrization²⁴

$$H_I = \sigma_I(\infty) + \frac{q_I}{\rho^2},$$

we observe that in [47] the constant terms are normalized to 1, which has the effect that the moduli dependence is scaled into the $\frac{1}{\rho^2}$ term. To understand

²³ The original notation in [47] is α, γ, σ , and Q_N is denoted N . Also note that in comparison to [47] $r_0^2 = 2c$.

²⁴ We now let the indices I take values $I = 1, 5, N$ instead of $1, 2, 3$.

the relation between the brane charges Q_I and our inner horizon charges q_I it is sufficient to set $\sigma_I(\infty) = 1$. Then

$$q_I = c \sinh^2(\alpha_I) ,$$

and using this we find:

$$Q_1 = 2 \frac{V}{g} \sqrt{q_I(q_I + c)} , \quad Q_5 = 2 \frac{1}{g} \sqrt{q_I(q_I + c)} , \quad Q_N = 2 \frac{R^2 V}{g^2} \sqrt{q_I(q_I + c)} . \quad (51)$$

Thus for fixed moduli V, R, g the charges Q_I and q_I are proportional, up to higher order terms in c . From the microscopic point of view it is natural to perform the extremal limit such that the integer valued charges Q_I are kept fixed. Then q_I and \tilde{q}_I are not constant, but the extra terms are subleading in c .

For completeness we mention that in the non-extremal case the integer valued charges do not count the total numbers of D1 branes, D5 branes and quanta of momentum, but the differences in the numbers of branes and anti-branes, and of left- and right moving momenta. Non-extremal black holes can be interpreted as systems of branes and anti-branes, and, surprisingly, the resulting formulae for mass and entropy look like those of a non-interacting system²⁵. It should be interesting to investigate whether the ‘dressed attractor mechanism’ described above can shed some light onto such systems and, possibly, onto their dynamical evolution towards the extremal limit.

3.3 The universal solution

Let us now return to the general class of models, where we do not make any additional assumptions about the scalar metric. We can still find a solution by imposing (46)

$$4c^2 A_J A_K - B_J B_K + \tilde{q}_J \tilde{q}_K = 0 ,$$

but in order to solve the original constraint (45) this must now hold for all values for J and K . Already the equations where $J = K$ fix n constants. For example we can solve for the B_J in terms of A_J and the charges \tilde{q}_I :

$$B_J = \sqrt{\tilde{q}_J^2 + 4c^2 A_J^2} .$$

The remaining equations, where $J \neq K$, can only be solved if we take $A_I \propto \tilde{q}_I$, which in turn implies that $B_I \propto \tilde{q}_I$. The possible solutions can be parametrized in the form

$$A_J = \mu \tilde{q}_J , \quad B_K = \tilde{q}_K \sqrt{1 + 4c^2 \mu^2} ,$$

where μ is a parameter which reflects that the overall normalization of A_J, B_K relative to the charges is not fixed by the constraint.

Writing the solution in the form

$$\sigma_I(\rho) = \frac{H_I(\rho)}{W^{1/2}(\rho)} , \quad H_I(\rho) = \sigma_I(\infty) + \frac{q_I}{\rho^2} ,$$

we find

$$A_I = \sigma_I(\infty) = \mu \tilde{q}_I , \quad q_I = \frac{1}{2} \tilde{q}_I \left(\sqrt{1 + 4c^2 \mu^2} - 2c\mu \right) .$$

²⁵Callan^{1996dv}, Horowitz:1996fn, Maldacena:1996ky

Therefore the harmonic functions H_I are proportional to one another,

$$H_1 \propto H_2 \propto \cdots \propto H_p \propto \mathcal{H} = 1 + \frac{q}{\rho^2} ,$$

and the solution can be expressed in terms of two independent functions $W(\rho)$ and $\mathcal{H}(\rho)$. This implies that the metric takes the Reissner-Nordstrom form, and that the five-dimensional scalars are constant. In the previous section we derived this for diagonal models, but it remains valid here because we only need to use the homogeneity properties of the scalar metric. Since all harmonic functions are proportional, we are effectively dealing with homogeneous functions of one variable, which are determined, up to overall normalization, by their degree.

In particular, the dual scalars σ_I are homogeneous functions of degree -1 of the scalars σ^I . Given that the universal solution takes the form $\sigma_I \propto \mathcal{H}W^{-1/2}$, it follows that $\sigma^I \propto \mathcal{H}^{-1}W^{1/2}$. The prepotential is homogeneous of degree p , and therefore

$$\hat{\mathcal{V}}(\sigma) \propto W^{p/2} \mathcal{H}^{-p} ,$$

which implies that the line element takes the Reissner-Nordstrom form. The five-dimensional scalars h^I are homogeneous of degree zero in the harmonic function, and therefore must be constant if the harmonic functions are proportional.

This also clarifies the role of the parameter μ . When lifting to five dimensions we impose the normalization condition

$$e^{\tilde{\sigma}(\rho)} = \frac{W^{1/2}}{\hat{\mathcal{V}}(H(\rho))^{1/p}} \xrightarrow{\rho \rightarrow \infty} 1 .$$

This is a condition on the asymptotic four-dimensional scalars $\sigma_I(\infty) = \mu \tilde{q}_I$, which for the universal solution are proportional to the charges. Therefore the parameter μ needs to be used to normalize the five-dimensional metric. In the four-dimensional set-up, μ is not fixed and encodes the relation between the five-dimensional and four-dimensional Newton constant.

3.4 Block-diagonal models

There are intermediate cases where the Christoffel symbols $\partial_I a^{JK}$ simultaneously assume block-diagonal form, or can be brought to this form, by a linear transformation. For concreteness, suppose that the indices split into two subsets

$$1 \leq J_1, K_1 \leq m , \quad m < J_2, K_2 \leq n ,$$

such that $\partial_I a^{I_1 J_2} = 0$ for all I . Then we obtain a solution of (45) by imposing (46) for $J = K$, $J_1 \neq K_1$ and $J_2 \neq K_2$, but we do not need to impose it if J and K belong to different blocks.

The ‘diagonal’ constraints imply

$$B_J = \sqrt{\tilde{q}_J + 4c^2 A_J^2} .$$

But since there are no ‘off-diagonal’ constraints if I and J belong to different blocks, we obtain

$$A_{J_k} = \mu_k \tilde{q}_{J_k} , \quad B_{J_k} = \tilde{q}_{J_k} \sqrt{1 + 4c^2 \mu_k^2} ,$$

where $k = 1, 2$. As a result only harmonic functions belonging to the same block must be proportional to one another:

$$H_1 \propto \cdots H_m \propto \mathcal{H}_1, \quad H_{m+1} \propto \cdots H_n \propto \mathcal{H}_2,$$

and the solution depends on three independent harmonic functions $W, \mathcal{H}_1, \mathcal{H}_2$. After lifting to five dimensions one combination of the parameters μ_1 and μ_2 is fixed by normalizing the metric at infinity. There remains one undetermined parameter which allows to vary the value of one five-dimensional scalar field at infinity.

For models with a larger number of blocks the number of undetermined moduli at infinity and hence of non-constant scalar fields increases. If the Christoffel symbols decompose into k blocks, then $k - 1$ five-dimensional scalars can be non-constant. While $k = 1$ corresponds to the universal solution, where all scalars are constant, $k = n$ corresponds to diagonal models, where all $n - 1$ five-dimensional scalars can be non-constant.

Block-diagonal Christoffel symbols with two blocks occur when the Hesse potential takes the form

$$\mathcal{V} = -\frac{1}{p} \log \left(\hat{\mathcal{V}}_1(\sigma^1, \dots, \sigma^m) \hat{\mathcal{V}}_2(\sigma^{m+1}, \dots, \sigma^n) \right),$$

where \mathcal{V}_1 and \mathcal{V}_2 are homogeneous functions of degrees r and s , where $r + s = p$. A higher number of blocks occurs when the Hesse potential factorizes into more homogeneous factors, and the extreme case of a diagonal model occurs for complete factorization into factors of degree one, $\hat{\mathcal{V}}_I \propto \sigma_I$.

Of course we expect that even for generic models solutions exist, where all scalars are non-constant, because such solutions exist in the extremal limit. However the solutions which we have constructed explicitly in this article only have a limited number of non-constant scalar fields. Metrics where the prepotential factorizes into independent homogeneous factors are in particular product metrics and therefore rather special. Thus it is important to make progress by finding more general solutions for models without block structure.

4 Conclusions and Outlook

In this paper we have demonstrated that non-extremal black hole solutions can be obtained from extremal ones by a universal deformation which is blind to the details of the matter sector. While the class of models for which explicit solutions were obtained happens to be based on symmetric spaces, the relevant features for obtaining solutions were given by the generalized special geometry, through the existence of a potential together with homogeneity properties. What played a crucial role, however, was the factorization of the target space into two-dimensional spaces with simple geodesics, as is clear from the fact that the number of explicit solutions that we could obtain is correlated with the number of blocks into which the scalar metric can be decomposed. Therefore we expect that further progress will require a more detailed understanding of geodesics in generalized special real manifolds. Since the general analysis of the field equations allows the presence of an extra term in the contracted scalar field equation (41), which vanishes for diagonal models, this term is likely to come

into play for non-diagonal models. It is encouraging that the geometry obtained by reducing the black hole space-time with respect to time, the time-reduced Reissner-Nordstrom metric, is completely fixed and independent of the matter sector. The other feature which we observed, and which works universally in diagonal models, is that the non-extremal solution is obtained by dressing metric and scalar fields by an additional harmonic function. Since this is closely related to the homogeneity properties of the scalar manifold, which also hold for non-diagonal models, we expect that progress can be made without assuming that the target space is a symmetric space. The problem of solving the field equations amounts to constructing a harmonic map from the reduced space time into the target space. For spherically symmetric solutions this reduces to constructing geodesic curves. The difference between extremal and non-extremal solutions is that the former correspond to null geodesics while the later ones correspond to space-like geodesics. A further difference, which is obscured by the reduction to the radial coordinate, but manifest as long as we only reduce over time, is that for extremal solutions the time-reduced geometry is flat, while for non-extremal solutions it is only conformally flat.²⁶ This shows how the harmonic map gets deformed when making solutions non-extremal: the geometry of the reduced space-time is modified by a conformal factor, which forces the geodesic to become non-null, and this manifests itself through the dressing of the solution by an additional harmonic function. Upon reduction to the radial coordinate the conformal factor of the reduced space-time becomes encoded in the relation between the standard radial coordinate ρ and the affine curve parameter τ of the geodesic. None of these observations are specific to diagonal models, and thus we expect that the general class of models can be understood by digging deeper into the geometry of the harmonic map.

It should also be instructive to relate our work to approaches based on first order flow equations and integrability [32, 33, 34, 35, 17, 36, 37, 38]. Flow equations and harmonic functions are intimately related. In [41] the reduction of the harmonic equation to a first order equation was shown to be the result of the existence of n conserved charges. While this was done for extremal solutions, only, the argument should carry over to the non-extremal solutions considered here, because in terms of the radial variable ρ the solution for the five-dimensional scalar fields remains the same. The non-extremal deformation is fully encoded in the modified relation between the radial variable ρ and the affine parameter τ . This is interesting, because the argument given in [41] does not require the target space to be symmetric, but only the existence of n isometries. The approach via symmetric spaces is closely related to integrability and the Hamilton-Jacobi formalism. The latter is used in order to identify adapted parametrizations of the field equations. Our approach uses geometrical considerations in order to arrive directly at such a parametrization, given by the dual scalar fields σ_I and the affine curve parameter τ . For extremal black holes this was briefly investigated in [41], and we plan to explore this more systematically in the future.

In this paper we have restricted ourselves to static, five-dimensional black holes. The extension to various other types of solutions should be interesting to investigate. Since supersymmetry does not play an immediate role, the adaptation of our results to dimensions other than four is straightforward and amounts

²⁶Here we use that any spherically symmetric metric can be brought to isotropic form [55].

to adjusting numerical factors. However, by working in five dimensions we have restricted ourselves to electric charges, while in four dimensions generic charged black holes carry both electric and magnetic charge. Applying dimensional reduction to this case leads to a more complicated target space geometry, with an isometry group which is solvable (of Heisenberg group type) rather than abelian, as is well known from the c-map [45, 54]. We believe that this is best approached systematically by re-visiting and generalizing the c-map, which we leave to future work. At the current stage we see no problem in principle, and expect that the features we have observed will pertain.

Other extensions would naturally include the study of rotating solutions, the addition of a cosmological constant, Taub-NUT charge (i.e. more complicated Ricci flat and conformally Ricci flat time-reduced geometries), black strings and black rings, domain walls and cosmological solutions. Of course there is already a large literature on all these types of solutions, and dimensional reduction is often used as one of the tools. For example, black ring solutions were constructed using reduction over time in [60]. However, we believe that dimensional reduction could play an even bigger role in particular in handling generic matter sectors and organizing solutions, if the underlying geometry of harmonic maps is fully exploited. Concerning cosmological solutions it is interesting that we found solutions which extend to the inner horizon, because the Killing vector becomes space-like between the horizons. Thus the scalar flow becomes a flow in time between the horizons. The between-horizon geometry of charged, non-extremal solution is a natural starting point for the construction of cosmological solutions of the S-brane type [51, 52]. Non-extremal black hole solutions can also be used to obtain ‘mirage-type’ cosmologies, where FRW cosmology is induced on branes moving in the black hole background [16].

It has been observed that in cases where a reduction of the field equations to first order flow equations takes place, there is a close relation between black holes and other types of solutions including domain walls, instantons and cosmologies. The frameworks proposed for capturing these relations are characterized by the key words ‘fake (super-)potentials’, ‘fake-’ or ‘pseudo-’Killing spinors and ‘fake supersymmetry’ [56, 57]. The ‘generalized special geometries’ used in this article are similar in spirit as they also aim to extend techniques originally developed within a supersymmetric set-up to more general non-supersymmetric situations. It should be interesting to explore the relations between these frameworks. We note that the reduction over time introduces ‘variant real forms’ of special geometry, specifically the Euclidean special geometries described in [58, 59, 46].²⁷ Similar observations have been made with regard to maximal supergravities, their toroidal reductions and the temporal T-dualization of type-II string theories [61, 62, 63, 64]. This indicates a unifying pattern underlying (super-)gravity solutions, branes, and their various mutual relations, which deserves further exploration.

²⁷The para-Kähler geometry of the extended scalar space obtained by reduction over time can be viewed as a generalization of the projective special para-Kähler geometry of Euclidean four-dimensional $N = 2$ vector multiplets constructed in [46].

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